RESEARCH STATEMENT

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Introduction

I am an algebraic topologist and my research is about the May spectral sequence and the Dyer-Lashof operations on Hopf rings.

One of the important goals in algebraic topology is to understand the stable homotopy groups of spheres $\pi_*^{st}(S)$. They have many connections to other areas of mathematics. One recent example is Wang and Xu's work [12] proving the uniqueness of the smooth structure on S^{61}. The recent work of Hill, Hopkins and Ravenel [2] on the Kervaire invariant one problem has renewed interest in these calculations, and work of Isaksen, Wang and Xu has pushed the calculations to the nineties stems. Despite this, the May spectral sequence has the potential to push the calculations far further. It is an effective way to compute the cohomology of the Steenrod algebra $\mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$ which is the E_2 page of the Adams spectral sequence which converges to the stable homotopy groups of spheres.

My work started with the E_2 page of the May spectral sequence at the prime 2. Compared to the cohomology of the Steenrod algebra, the E_2 page of the May spectral sequence can be computed in a much bigger range. May [5] conjectured what all the indecomposables of the E_2 page are and I have conjectured all the relations among these indecomposables (See [4]). I have proven both conjectures in a large range of dimensions, which indicates that it is possible that these indecomposables and relations do in fact describe the whole E_2 page. I also show that the E_2 page is nilpotent free in this range, and I conjecture that the whole E_2 page is nilpotent free. This is startling because all of the elements in positive dimensions in the stable homotopy groups of spheres are nilpotent. Apart from being a tool to compute the cohomology of the Steenrod algebra, the E_2 page of the May spectral sequence has strong connections to Massey products, which makes it interesting in its own right.

My other project is about the Dyer-Lashof operations on Hopf rings. Hopf rings concern the homology of all of the spaces R_n of a commutative ring spectrum R. In 2017, Tyler Lawson proved that the Brown-Peterson spectrum at the prime 2 is not an E_∞ ring spectrum. He used secondary power operations and his proof involved partial information about Dyer-Lashof operations on Hopf rings (see [11]). It is known by Cohen, Lada and May [1] that the homology of the zeroth space R_0 of an E_∞-ring spectra R is equipped with two kinds of Dyer-Lashof operations, one additive and one multiplicative, and the homology of the spectrum R also has Dyer-Lashof operations. However, the two kinds of operations on the Hopf rings have hardly been studied at all. An appropriate definition of the multiplicative operations on the homology Hopf rings is still lacking. I have defined such multiplicative operations on $H_*(R_n)$ for varying n and they converge to the Dyer-Lashof operations on $H_*(R)$.

Project 1: The May Spectral Sequence

Problem 1. The generators and relations of E_2. Let $\mathscr{A}$ be the Steenrod algebra at the prime 2. We have the following May spectral sequence:

$$E_2 = \operatorname{Ext}^{s,t,p}_{E^0\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \operatorname{Ext}^{s,t}_{\mathscr{A}}(\mathbb{F}_2, \mathbb{F}_2).$$

May has computed the E_2-term in stems $t-s \leq 164$. He has defined the elements h_i, b^i_j and $h_i(S)$ in E_2 and proved that they are indecomposables. The following conjecture is at least true in stems $t-s \leq 164$ and could be true in general.

Conjecture 1 (May)**.** *The elements h_i, $b^i_j (j > 1)$ and $h_i(S)$ form a basis for the indecomposables of E_2.*

In addition to this conjecture, I made the following two conjectures about the relations in E_2.

Conjecture 2 (Lin, [4])**.** *The relations in E_2 are implied by the following. Each of them needs some conditions on i, j, S, T, etc.*

(1) $\sum_k b_{ik}b_{kj} = 0$
(2) $h_{S_1,T_1}h_{S_2,T_2} = 0$
(3) $\sum_{s\in S} b_{sj}h_{S-\{s\},T+\{s\}} = 0$
(4) $\sum_{t\in T} b_{it}h_{S+\{t\},T-\{t\}} = 0$
(5) $h_{S_1,T_1}h_{S_2,T_2} = \sum_{I\subset T''_1\cap S_2} h_{S''_1+I,T''_1-I}h_{S'_1+S_2-I,T'_1+T_2+I}$
(6) $h_{S_1,T_1}h_{S_2,T_2} = \sum_{I\subset T_1\cap S''_2} h_{S''_2-I,T''_2+I}h_{S_1+S'_2+I,T_1+T'_2-I}$
(7) $h_{S_1,T_1}h_{S_2,T_2} = \sum_{\substack{I\subset S'_1\\ J\subset T'_2}} h_{S'_1-I,T'_1+I}h_{S'_2+J,T'_1-J}b_{S''_1+I,T''_2+J}$
(8) *If $\sum_i x_ih_{S_i-\{a\},T_i-\{b\}} = 0$ and each x_i satisfies some conditions, then*
$$\sum_i x_ih_{S_i,T_i} = 0.$$

Here b_{ij} is my notation for b^i_{j-i}, $h_{S,T}$ is equal to some product of $h_i(S')$ and $b_{S,T}$ is a polynomial in the b_{ij}.

Conjecture 3 (Lin, [4])**.** *The algebra E_2 is nilpotent free.*

One support for Conjecture 2 is the following theorem.

Theorem 4 (Lin, [4])**.** *The relations (1)-(6) in Conjecture 2 hold in all degrees while relations (7), (8) hold at least in the range $t-s \leq 285$.*

The algebra E_2 has a big subalgebra HX_7 which is the E_2-term of the May spectral sequence for $\mathscr{A}_6$, i.e., we have a spectral sequence

$$HX_7 \Longrightarrow \operatorname{Ext}^{*,*}_{\mathscr{A}_6}(\mathbb{F}_2, \mathbb{F}_2).$$

Here $\mathscr{A}_6$ is the subalgebra of the Steenrod algebra generated by Sq^{2^i}, $0 \leq i \leq 6$. The subalgebra HX_7 covers a large range of dimensions in E_2. We have a homomorphism

$$\frac{HX_7 \otimes \mathbb{F}_2[h_7]}{(h_6h_7, h_4(1)h_7)} \longrightarrow E_2$$

which is isomorphic in the range $t-s \leq 285$.

With the assistance of Gröbner bases and computer programming, I computed HX_7 and the result shows the following.

Theorem 5 (Lin, [4]). *Conjectures 1 and 2 hold in HX_7.*

The tool I used to describe an additive basis of HX_7 is the theory of Gröbner bases. The use of Gröbner bases led me to the following unexpected discovery.

Theorem 6 (Lin, [4]). *The subalgebra HX_7 of E_2 is nilpotent free.*

These theorems are strong evidence for Conjectures 1, 2 and 3 since now we know that they are all true at least in the range $t - s \leq 285$.

Problem 2. Connection to Massey products. The E_2 page of the May spectral sequence is isomorphic to HX where X is a polynomial differential graded algebra given by the following

$$X = \mathbb{F}_2[R_{ij} : 0 \leq i < j],$$
$$dR_{ij} = \sum_k R_{ik} R_{kj}.$$

We define the subalgebra $X_n = \mathbb{F}_2[R_{ij} : 0 \leq i < j \leq n]$. I observed that the differential algebra X_n plays an important role in Massey products.

Theorem 7 (Lin, [4]). *If A is a commutative differential algebra, then the decompositions of zero in HA as an n-ary Massey product (together with a defining system)*

$$0 \in \langle a_1, \ldots, a_n \rangle, \qquad a_i \in HA$$

are in one-to-one correspondence to maps of differential algebras:

$$f : X_n \to A$$

where f induces the algebraic map

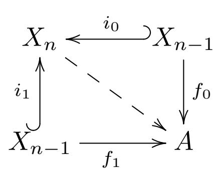
$$f_* : HX_n \to HA$$

with $f_*(h_{i-1}) = a_i$, $1 \leq i \leq n$.

Theorem 8 (Lin, [4]). *A nontrivial element $a \in HA$ and a defining system for the Massey product*

$$a \in \langle a_1, \ldots, a_n \rangle$$

corresponds to the obstruction to obtaining the dashed map



where f_0 corresponds to the sub-defining system for $0 \in \langle a_1, \ldots, a_{n-1} \rangle$ and f_1 for $0 \in \langle a_2, \ldots, a_n \rangle$. The embeddings i_0 and i_1 are given by $i_0(R_{ij}) = R_{ij}$ and $i_1(R_{ij}) = R_{i+1,j+1}$.

The E_2 page of the May spectral sequence itself has a lot of interesting Massey products.

Theorem 9 (Lin-May, [4]). *In E_2, we have*

$$h_i(S) \in \langle h_i, h_{i+1}, \ldots, h_{i+2n-2}, h_i'(S) \rangle.$$

Here $h_i'(S)$ is a product of some indecomposables $h_j(S_1)$ determined by $h_i(S)$.

The proof was done by May in the past in his unpublished work and I give my own proof in [4]. I also show that there is no indeterminacy at least in the range $t-s\leq 285$ with the assistance of computer programs.

Problem 3. Computation of the May spectral sequence. In the E_2 page, the d_2 differentials of b_{ij} are computed by May:

$$d_2(b_{i,i+2}) = h_{i+1}^3 + h_i^2 h_{i+2},$$

$$d_2(b_{ij}) = h_{i+1}b_{i+1,j} + b_{i,j-1}h_{j+1},\ j-i>2.$$

May also computed $d_2(h_i(S))$ in the range $t-s\leq 164$.

- $d_2(h_i)=0$,
- $d_2(h_i(1)) = h_i h_{i+2}^2$,
- $d_2(h_i(1,3)) = h_i h_{i+2} h_{i+2}(1) + h_i(1)h_{i+4}^2$,
- $d_2(h_i(1,2)) = h_{i+3}h_i(1,3)$.

By doing an explicit construction in the cobar complex of $\mathscr{A}_*$, I am able to prove the following in all stems

Theorem 10 (Lin, [4])**.** *The d_2 differential on $h_i(S)$ is given by the following:*

$$d_2 h_i(s_1,\ldots,s_{n-1}) = \sum_{\substack{j=n-1 \ or \\ s_j+1<s_{j+1}}} h_{i+s_j+1}h_i(s_1,\ldots,s_{j-1},s_j+1,s_{j+1},\ldots,s_{n-1}).$$

To do further computations in the May spectral sequence beyond the E_2 page, I have the following theorem which systematically extends one of the techniques used by May and Tangora.

Theorem 11 (Lin, [4])**.** *Consider the $\mathscr{A}_*$-free coresolution $\tilde{C}(\mathscr{A}_*) = C(\mathscr{A}_*)\otimes\mathscr{A}_*$ of $\mathbb{F}_2$. There is a filtration on $\tilde{C}(\mathscr{A}_*)$ which is compatible with the May filtration on the cobar complex $C(\mathscr{A}_*)$ where $C_s(\mathscr{A}_*) = I(\mathscr{A}_*)^{\otimes s}$. The map $\varphi: C(\mathscr{A})\to\tilde{C}(\mathscr{A})$ induces a comparison map φ of spectral sequences. Consider the following composition of comparison maps of spectral sequences*

$$\begin{array}{ccc} E_2 & \Longrightarrow & \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2,\mathbb{F}_2) \\ \downarrow{\scriptstyle Sq^0} & & \downarrow{\scriptstyle Sq^0} \\ E_2 & \Longrightarrow & \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2,\mathbb{F}_2) \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ \tilde{E}_2 & \Longrightarrow & \mathbb{F}_2. \end{array}$$

The composition in E_2-terms

$$E_2 \xrightarrow{Sq^0} E_2 \xrightarrow{\varphi} \tilde{E}_2$$

is injective.

This theorem is useful because compared to $E_2 \Longrightarrow \mathrm{Ext}_{\mathscr{A}}^{*,*}(\mathbb{F}_2,\mathbb{F}_2)$, the spectral sequence $\tilde{E}_2\Longrightarrow\mathbb{F}_2$ is much easier to compute due to the fact that everything in positive degrees has to be killed by differentials.

PROJECT 2: THE DYER-LASHOF OPERATIONS ON HOPF RINGS

Problem 4. Define multiplicative operations on Hopf rings. Let R be an E_∞-ring spectrum. Consider the Hopf ring $\bigoplus_n H_*(R_n)$ where R_n is the n'th space of R. We know that each R_n is an E_∞ space and R_0 is an E_∞ ring space. We have the following Dyer-Lashof operations:

$$Q^s : H_k(R_n) \to H_{k+s}(R_n), \quad \text{for all } n,$$
$$\tilde{Q}^s : H_k(R_0) \to H_{k+s}(R_0).$$

where Q^s are the additive Dyer-Lashof operations and $\tilde{Q}^s$ are the multiplicative Dyer-Lashof operations.

We know that $\Sigma^\infty R_0 \to R$ is a map of E_∞-spectra and the multiplicative operations $\tilde{Q}^s$ correspond to the Dyer-Lashof operations on $H_*(R)$. However, the $\tilde{Q}^s$ cannot recover the Dyer-Lashof operations on $H_*(R)$ since they only act on $H_*(R_0)$. In order to extend $\tilde{Q}^s$ to multiplicative operations on $\bigoplus_n H_*(R_n)$ I define the following:

Definition 12. A collection of spaces $\{\mathscr{G}(k,m,n)\}_{k\geq 0, m>0, n\geq mk}$ is called a Hopf operad if Σ_k acts on $\mathscr{G}(k,m,n)$ and there is a structural map

$$\gamma : \mathscr{G}(k,n,l) \times \mathscr{G}(j_1,m,n) \times \cdots \times \mathscr{G}(j_k,m,n) \longrightarrow \mathscr{G}(j_1+\cdots+j_k,m,l)$$

with axioms similar to those of an operad (See [1]).

Definition 13. We say that $(\mathscr{G},\mathscr{C})$ is a Hopf operad pair if $\mathscr{C}$ is an operad, $\mathscr{G}$ is a Hopf operad and there is an action

$$\lambda : \mathscr{G}(k,m,n) \times \mathscr{C}(j_1) \times \cdots \times \mathscr{C}(j_k) \longrightarrow \mathscr{C}(j_1 \cdots j_k)$$

with axioms similar to those of an operad pair.

Definition 14. A $(\mathscr{G},\mathscr{C})$-spectrum R is a spectrum with actions

$$\theta_j : \mathscr{C}(j) \times R_n^j \to R_n,$$
$$\xi_k : \mathscr{G}(k,n,m) \times R_n^k \to R_m$$

and axioms including that the following distributivity diagram commutes:

$$\begin{array}{ccc}
\mathscr{G}(k,m,n)\times\mathscr{C}(j_1)\times R_m^{j_1}\times\cdots\times\mathscr{C}(j_k)\times R_m^{j_k} & \xrightarrow{1\times\theta_{j_1}\times\cdots\times\theta_{j_k}} & \mathscr{G}(k,m,n)\times R_m^k \\
\downarrow{\scriptstyle \xi_k} & & \downarrow{\scriptstyle \xi_k} \\
\mathscr{C}(j)\times R_n^j & \xrightarrow[\theta_j]{} & R_n
\end{array}$$

Theorem 15 (Lin)**.** *If R is an E_∞ ring spectrum with action by the linear isometries operad, then there exists an Hopf operad pair $(\mathscr{G},\mathscr{C})$ which acts on R, where $\mathscr{C}$ is an E_∞-operad and $\mathscr{G}(k,m,n)$ is a Thom space of an $(n-mk)$-dimensional vector bundle over an $(n-mk-1)$-connected space.*

Definition 16. If the pair $(\mathscr{G},\mathscr{C})$ acts on R, then for any prime p we can define

$$\tilde{Q}_{n-pm,i} : H_k(R_m) \to H_{pk+i+n-pm}(R_n)$$

where $i \leq n-pm$, $\tilde{Q}_{n-pm,i}(x) = (\xi_k)_*(\phi(e_i)\otimes x^p)$ and ϕ is the Thom isomorphism.

The above operations converge to the Dyer-Lashof operations on the spectra level when we fix $n-pm$ and let m go to infinity:

Theorem 17 (Lin)**.** *The following diagram commutes:*

$$\begin{array}{ccc} H_k(R_m) & \xrightarrow{\tilde{Q}_{n-pm,i}} & H_{pk+i+n-pm}(R_n) \\ \sigma_* \downarrow & & \downarrow \sigma_* \\ H_{k-m}(R) & \xrightarrow[Q_i]{} & H_{pk+i-pm}(R) \end{array}.$$

where σ is induced by $\Sigma^\infty R_n \to \Sigma^{-n} R$ and $Q_i(x) = \theta_*(e_i \otimes x^p)$ for $H_*(R)$. The relationship between Q^s and Q_i can be found in [1].

Current and Future Work

Problem 5. The cohomology of the Steenrod algebra. The May spectral sequence is a powerful tool when one tries to analyze the cohomology of the Steenrod algebra. Because the E_2 page of the May spectral sequence is a tri-graded commutative algebra, it can take advantage of the parallel computing capability of modern computers. It also helps in computing the Massey products in the cohomology of the Steenrod algebra especially when there is no extension or indeterminacy for degree reasons.

One of my ongoing projects is to compute the May spectral sequence for each $t = 1, 2, \ldots$ Currently for the E_2 term I have results up to $t = 600$ and for E_4 up to $t = 210$, both obtained with help of computer programs. I am also trying to compute the May spectral sequence for $s = 1, 2, \ldots$. The cohomology of the Steenrod algebra $\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ is currently known for $s \leq 4$ and partially known for $s = 5$ using the lambda algebra (see [3]). The May spectral sequence together with Theorem 11 is very effective in computing $\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ by s because the power operation Sq^0 preserves s so that one can iterate Sq^0 in Theorem 11. Combining my methods with others, we might be able to substantially extend the computations of the stable homotopy groups of spheres.

Other projects include the computation of $\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ modulo nilpotent elements based on the work of Palmieri [7] and the computation of the stable Sq^0-families in $\mathrm{Ext}_{\mathscr{A}}^{s,t}(\mathbb{F}_2, \mathbb{F}_2)$ by the perf version of the May spectral sequence

$$\mathrm{Ext}_{E^0\mathscr{A}^{perf}}^{***}(\mathbb{F}_2, \mathbb{F}_2) \Longrightarrow \mathrm{Ext}_{\mathscr{A}^{perf}}^{**}(\mathbb{F}_2, \mathbb{F}_2).$$

Problem 6: The properties of multiplicative operations on Hopf rings. Although I have set up the definition of the multiplicative operations on Hopf rings of E_∞ ring spectra and shown that it can recover the Dyer-Lashof operations on the Hopf rings homology of spectra, these operations should satisfy properties analogous to those of the usual Dyer-Lashof operations including the Cartan formula, the Adem relations, the Nishida relations, the mixed Cartan relations and the mixed Adem relations, etc. One of my plans is to verify those formulas. I also want to make calculations of these operations on the Hopf rings of KU and MU, etc.

References

[1] Frederick R. Cohen, Thomas J. Lada, and J. Peter May, *The homology of iterated loop spaces*, Lecture Notes in Mathematics, Vol. 533, Springer-Verlag, Berlin-New York, 1976.

[2] Michael A. Hill, Michael J.Hopkins, Douglas C. Ravenel, *On the non-existence of elements of Kervaire invariant one.* Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. II, 1219–1243. Kyung Moon Sa, Seoul, 2014.

[3] W.H. Lin, $\mathrm{Ext}_{\mathscr{A}}^{4,*}(\mathbb{F}_2, \mathbb{F}_2)$ *and* $\mathrm{Ext}_{\mathscr{A}}^{5,*}(\mathbb{F}_2, \mathbb{F}_2)$, Topology Appl., 155 (2008), pp. 459-496

[4] Weinan Lin, *On the May spectral sequence at the prime 2* (2020), preprint, arXiv:2010.14754
[5] J. Peter May, *The Cohomology of Restricted Lie Algebras and of Hopf Algebras: Application to the Steenrod Algebra*, Thesis (Ph.D.)–Princeton University, 1964.
[6] J. Peter May, *Matric Massey products*, Journal of algebra 12, 533-568 (1969)
[7] John H, Palmieri, *Quillen stratification for the Steenrod algebra.* Ann. of Math. (2) 149 (1999), no. 2, 421–449.
[8] D.C. Ravenel, *Complex cobordism and stable homotopy groups of spheres*, Pure and Applied Mathematics, Pure and Applied Mathematics, 121. Academic Press, Inc., Orlando, FL, 1986.
[9] D.C. Ravenel and W.S. Wilson, *The Hopf ring for complex cobordism*, J. Pure Appl. Algebra, 9:241–280, 1977.
[10] M.C. Tangora. *On the cohomology of the Steenrod algebra*, Math. Z. 116 (1970), 18–64.
[11] Tyler Lawson, *Secondary power operations and the Brown-Peterson spectrum at the prime 2*, Ann. of Math. (2) 188 (2018), no. 2, 513–576.
[12] Guozhen Wang and Zhouli Xu, *Some extensions in the Adams spectral sequence and the 51-stem*, Algebr. Geom. Topol. 18 (2018), no. 7, 3887–3906.