LECTURE 3 - SIMPLICIAL SETS

Definition 1. A *simplicial set* K is a sequence of sets K_n, $n \geq 0$, and functions $d_i : K_n \to K_{n-1}$ and $s_i : K_n \to K_{n+1}$ for $0 \leq i \leq n$ that satisfies

$$d_i \circ d_j = d_{j-1} \circ d_i, \text{ if } i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \\ id, & \text{if } i = j \text{ or } i = j+1 \\ s_j \circ d_{i-1}, & \text{if } i > j+1 \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i, \text{ if } i \leq j.$$

We define the category Δ of finite ordered sets.

Definition 2. The objects of Δ are the finite ordered set $[n] = \{0, \dots, n\}$. Its morphisms are the nondecreasing functions $\mu : [m] \to [n]$. Define particular nondecreasing functions

$$\delta_i : [n-1] \to [n] \text{ and } \sigma_i : [n+1] \to [n]$$

for $0 \leq i \leq n$ by

$$\delta_i(j) = \begin{cases} j & \text{if } j < i \\ j+1 & \text{if } j \geq i \end{cases}$$

and

$$\sigma_i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i. \end{cases}$$

In other words, δ_i skips i and σ repeats i.

Proposition 3. *Every nondecreasing function $\mu : [m] \to [n]$ can be written as a composite of δ_i and σ_j for varying i and j.*

Proposition 4. *The category of simplicial sets can be identified with the category of (covariant) functors*

$$K : \Delta^{op} \to \mathscr{S}et$$

and natural transformations between them.

Definition 5. A *simplicial object* in a category $\mathscr{C}$ is a contravariant functor $K : \Delta \to \mathscr{C}$. These functors and natural transformations between them forms the simplicial category $s\mathscr{C}$. Any functor $F : \mathscr{C} \to \mathscr{D}$ induces a functor $sF : \mathscr{C} \to \mathscr{D}$.

Dually, a covariant functor $\Delta \to \mathscr{C}$ is called a *cosimplicial object* in $\mathscr{C}$.

We have "standard simplices" in many categories, including topological spaces, simplicial sets, and even posets and categories. Those can be encoded by a standard cosimplicial object in $\mathscr{V}$, written by a covariant functor

$$\Delta[\bullet]^v : \Delta \to \mathscr{V}.$$

The superscript v is used to distinguish these standard cosimplicial objects in different categories.

For each object V in $\mathscr{V}$, we obtain a contravariant functor, denoted $SV : \Delta \to \mathscr{S}et$, by letting the set S_nV of n-simplices be the set $\mathscr{V}(\Delta[n]^v, V)$. In other words, we have a functor

$$S : \mathscr{V} \to s\mathscr{S}et. \tag{1}$$

Example 6. When $\mathscr{V} = \mathscr{U}$ is the category of topological spaces, then the functor S is exactly the singular complex. We construct $S_nX = \mathrm{Map}(\Delta[n]^t, X)$ where $\Delta[n]^t$ is the standard topological n-simplex.

Now we consider the case $\mathscr{V} = s\mathscr{S}et$.

Definition 7. Define the standard simplicial n-simplex $\Delta[n]^s$ to be the contravariant functor $\Delta \to s\mathscr{S}et$ represented by $[n]$. This means that the set $\Delta[n]^s_q$ of q-simplices is

$$\Delta[n]^s_q = \Delta([q], [n]).$$

The object $\Delta[\bullet]^s$ is a cosimplicial simplicial set, that is, a cosimplicial object in the category of simplicial sets.

Proposition 8. *Let K be a simplicial set. For $x \in K_n$, there is a unique map of simplicial sets $Y(x) : \Delta[n]^s \to K$ such that $Y(x)(\iota_n) = x$. Therefore*

$$K_n \cong s\mathscr{S}et(\Delta[n]^s, K).$$

Proof. This is a direct application of the Yoneda lemma. □

Next we consider the case $\mathscr{V} = s\mathscr{C}at$ and define the Nerve of a category.

Note that a poset can be viewed as a category with at most one arrow between any pair of objects: either $x \le y$ and then there is a unique arrow $x \to y$, or $x \not\le y$ and then there is no arrow from x to y. We can use this fact to define the standard cosimplicial object in $s\mathscr{C}at$.

Definition 9. We define a covariant functor

$$\Delta[\bullet]^c : \Delta \to s\mathscr{C}at$$

by sending the ordered set $[n]$ to the corresponding category $[n]$ and sending a morphism $\mu : [m] \to [n]$ to the corresponding functor $\mu_* : [m] \to [n]$. Thus $\Delta[\bullet]^c$ is a cosimplicial category.

We use this cosimplicial category and apply (1) to construct the nerve of a category.

Definition 10. Let $\mathscr{C}$ be a small category. We define a simplicial set $N\mathscr{C}$, called the nerve of $\mathscr{C}$. Its set $N_n\mathscr{C}$ of n-simplices is the set of covariant functors $\phi : [n]^c \to \mathscr{C}$. The function $\mu^* : N_n\mathscr{C} \to N_m\mathscr{C}$ induced by $\mu : [m] \to [n]$ is given by $\mu^*(\phi) = \phi \circ \mu_*$.

The definition can easily be unraveled. The vertices of $N_0\mathscr{C}$ is the set of objects of $\mathscr{C}$. An n-simplex is a choice of n composable morphisms

$$c_0 \xrightarrow{f_0} c_1 \xrightarrow{f_1} \cdots \xrightarrow{f_n} c_n.$$

The faces and degeneracies are given by

$$d_i(f_1, \dots, f_n) = (f_1, \dots, f_{i-1}, f_{i+1} \circ f_i, f_{i+2}, \dots, f_n)$$

and

$$s_i(f_1, \ldots, f_n) = (f_1, \ldots, f_{i-1}, id, f_i, \ldots, f_n)$$

Some authors may choose to reverse the arrows to define the nerve so that we can write $f_i \circ f_{i+1}$ instead of $f_{i+1} \circ f_i$.

The following example is very important.

Definition 11. Let G be a group regraded as a category with a single object $*$ and $Hom(*, *) = G$. The nerve NG is often written as B_*G and called the bar construction. It is the simplicial set with $B_nG = G^n$, with n-tuples of elements written as $[g_1|\cdots|g_n]$.