LECTURE 6 - CYCLIC HOMOLOGY

Cyclic sets.

Definition 1. A *cyclic object* A in a category $\mathscr{A}$ is a simplicial object together with an automorphism t_n of order $n+1$ on each A_n such that

$$d_i t = t d_{i-1},\ s_i t = t s_{i-1},\ i \neq 0$$

$$d_0 t_n = d_n, s_0 t_n = t_{n+1}^2 s_n.$$

We denote the category of cyclic objects by $s\mathscr{A}^c$. There a forgetful functor $j : s\mathscr{A}^c \to s\mathscr{A}$ from cyclic sets to simplicial sets.

Example 2. The simplicial set BG for a group G can be considered as a cyclic set by defining

$$t[g_1|\cdots|g_n] = [(g_1\cdots g_n)^{-1}|g_1|\ldots|g_{n-1}].$$

For $E_\bullet G = B_\bullet(G,G,e) = G^{\bullet+1}$, we define

$$t(g_0[g_1|\cdots|g_n]) = g_n[g_0|\cdots|g_{n-1}].$$

It is not hard to see that there is a natural inclusion $BG \subset EG$ as cyclic sets. (Check this!)

There is an equivalent definition of cyclic sets (See [1]), which is sometimes more convenient to work with.

Definition 3. A cyclic object A is a sequence of objects A_n and morphisms

$$d_i : A_n \to A_{n-1},\ 0 \leq i \leq n$$

$$s_i : A_n \to A_{n+1},\ 0 \leq i \leq n+1$$

such that

$$d_i \circ d_j = d_{j-1} \circ d_i,\ \text{if } i < j$$

$$d_i \circ s_j = \begin{cases} s_{j-1} \circ d_i & \text{if } i < j \text{ and } j - i \leq n \\ id, & \text{if } i = j \text{ or } i = j+1 \\ s_j \circ d_{i-1}, & \text{if } i > j+1 \end{cases}$$

$$s_i \circ s_j = s_{j+1} \circ s_i,\ \text{if } i \leq j.$$

$$(d_0 s_{n+1})^{n+1} = id : A_n \to A_n$$

Remark 4. The two equivalent definitions are linked by $t_n = d_0 s_{n+1}$ and $s_{n+1} = t_n s_n t_n^{-1}$.

We can formally define the category Λ^{op} by objects $[0], [1], [2], \ldots$ and generating morphisms

$$d_i : [n] \to [n-1],\ 0 \leq i \leq n$$

$$s_i : [n] \to [n+1],\ 0 \leq i \leq n+1$$

subject to the relations in Definition 3. In the opposite category Λ, d_i is written as δ_i and s_i is written as σ_i. The reader can find an alternate description of the category Λ in [2, Definition 9.6.3]. A cyclic object can be identified with a functor

$$A : \Lambda^{op} \to \mathscr{A}.$$

Definition 5. A *standard cyclic set* $\Lambda[n]$ is defined by

$$\Lambda[n]_q = \Lambda([q],[n]).$$

By Yoneda lemma we have

$$s\mathscr{S}et^c(\Lambda[n], A) \cong A_n.$$

Homework 6. Give an explicit description of the underlying simplicial set $j\Lambda[0]$ of the standard cyclic set $\Lambda[0]$. Prove that the geometric realization $|j\Lambda[0]|$ is homeomorphic to the circle S^1.

In fact, we have $|j\Lambda[n]| \cong S^1 \times |\Delta[n]|$ in general. (See [1, Proposition 2.7]) The geometric realization of the underlying simplicial set of a cyclic set A can be expressed as

$$|jA| \cong \coprod A_n \times |j\Lambda[n]| \Big/ \sim \tag{1}$$

$$(d_i a, x) \sim (a, \delta_i x),\ 0 \le i \le n$$

$$(s_i a, x) \sim (a, \sigma_i x),\ 0 \le i \le n+1$$

Here we are using definition 3 for the cyclic sets. The immediate consequence of (1) is that $|jA|$ is a space with an $S^1 \cong SO(2)$ action. Let $\mathscr{U}^c$ be the category of S^1-spaces. When both $s\mathscr{S}et^c$ and $\mathscr{U}^c$ are equipped with suitable model structures (cofibrations, fibrations and weak equivalences), we have the following.

Theorem 7 ([1, Theorem 4.2])**.** *$s\mathscr{S}et^c$ and $\mathscr{U}^c$ are Quillen equivalent.*

This implies that their homotopy categories are equivalent.

Cyclic Homology.

Assume that A is a cyclic object in an abelian category, the following double complex is called Tsygan's double complex and denoted by $CC_{**}(A)$.

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow b & & \downarrow b' & & \downarrow b' & & \downarrow b & & \\
A_4 & \xleftarrow{1+t} & A_4 & \xleftarrow{N} & A_4 & \xleftarrow{1+t} & A_4 & \xleftarrow{N} & \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
A_2 & \xleftarrow{1-t} & A_2 & \xleftarrow{N} & A_2 & \xleftarrow{1-t} & A_2 & \xleftarrow{N} & \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
A_1 & \xleftarrow{1+t} & A_1 & \xleftarrow{N} & A_1 & \xleftarrow{1+t} & A_1 & \xleftarrow{N} & \cdots \\
\downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \\
A_0 & \xleftarrow{1-t} & A_0 & \xleftarrow{N} & A_0 & \xleftarrow{1-t} & A_0 & \xleftarrow{N} & \cdots
\end{array}$$

Here $b = \sum_{i=0}^{n}(-1)^i d_i$ and $b' = \sum_{i=0}^{n-1}(-1)^i d_i$. We leave the reader to verify that this is indeed a double complex and the odd columns with b' as differentials are acyclic complexes.

Definition 8. The *cyclic homology* $\mathrm{HC}_*(A)$ for a cyclic object A in an abelian category is the homology of $\mathrm{Tot}\, CC_{**}(A)$. The cyclic homology of a k-algebra R is the cyclic homology of the cyclic sets $R \otimes R^{\otimes *}$.

Let CC^{01} be the first two columns of $CC_{**}(A)$, which is a double complex with other columns set to be trivial. The quotient $CC[-2] = CC/CC^{01}$ is a translation of CC and consider the short exact sequence of double complexes.

$$0 \to CC^{01} \xrightarrow{I} CC(A) \xrightarrow{S} CC[-2] \to 0.$$

The resulting long exact sequence is called "SBI" sequence.

Theorem 9 (SBI sequence)**.** *For any cyclic object A there is a long exact SBI sequence*

$$\cdots \mathrm{HC}_{n+1}(A) \xrightarrow{S} \mathrm{HC}_{n-1}(A) \xrightarrow{B} \mathrm{HH}_n(A) \xrightarrow{I} \mathrm{HC}_n(A) \xrightarrow{S} \mathrm{HC}_{n-2}(A) \cdots$$

Exercise 10. Compute $\mathrm{HC}_*(k[x])$.

If we extend Tsygan's first quadrant double complex to the left and obtain an upper half-plane double complex $CC^P_{**}(A)$, we can define the periodic cyclic homology.

$$\mathrm{HP}_*(A) = H_* \mathrm{Tot}^{\prod}(CC^P_{**}(A)).$$

We can also consider the negative subcomplex $CC^-_{**}(A)$ with columns $p \leq 0$ and define

$$\mathrm{HC}^-_*(A) = H_* \mathrm{Tot}^{\prod}(CC^-_{**}(A)).$$

If we filter Tsygan's double complex by columns we obtain a spectral sequence

$$E^2_{2p,q} = \mathrm{HH}_q(A) \Longrightarrow \mathrm{HC}_{2p+q}(A)$$

The d_2 differential $\mathrm{HH}_q(A) \to \mathrm{HH}_{q+1}(A)$ is called Connes' operator B.

Proposition 11. *The following square commutes*

$$\begin{array}{ccc} \Omega^n_{R/k} & \xrightarrow{\psi} & \mathrm{HH}_n(R) \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle B} \\ \Omega^{n+1}_{R/k} & \xrightarrow{\psi} & \mathrm{HH}_{n+1}(R) \end{array}$$

Theorem 12. *If R is a smooth commutative algebra and essentially of finite type over a field k of characteristic 0, then*

$$\mathrm{HC}_n(R) \cong \Omega^n_{R/k}/d\Omega^{n-1}_{R/k} \oplus H^{n-2}_{dR}(R) \oplus H^{n-2}_{dR}(R) \oplus \cdots$$

$$\mathrm{HP}_n(R) \cong \prod_{i \in \mathbb{Z}} H^{n+2i}_{dR}(R)$$

References

[1] W. G. Dwyer, M. J. Hopkins, and D. M. Kan. The homotopy theory of cyclic sets. *Trans. Amer. Math. Soc.*, 291(1):281–289, 1985.

[2] Charles A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1994.