LECTURE 5 - HOCHSCHILD HOMOLOGY

In this lecture we fix a commutative ring k and $\otimes = \otimes_k$.

Definition 1. Let R a k-algebra and M an R-R bimodule. We define a simplicial k-module $M \otimes R^{\otimes \bullet}$ with $[n] \mapsto M \otimes R^{\otimes n}$ by

$$d_i(m[r_1|\cdots|r_n]) = \begin{cases} mr_1[r_2|\cdots|r_n] & \text{if } i = 0 \\ m[r_1|\cdots|r_ir_{i+1}|\cdots|r_n] & \text{if } 0 < i < n \\ r_nm[r_1|\cdots|r_{n-1}] & \text{if } i = n \end{cases}$$

$$s_i(m[r_1|\cdots|r_n]) = m[r_1|\cdots|r_i|1|r_{i+1}|\cdots|r_n].$$

The *Hochschild homology* $\mathrm{HH}_*(R, M)$ of R with coefficients in M is defined to be the k-modules

$$\mathrm{HH}_n(R, M) = H_nC(M \otimes R^{\otimes *}),$$

where $C(M \otimes R^{\otimes *})$ is the associated chain complex with $d = \sum (-1)^i d_i$.

Proposition 2. $\mathrm{HH}_0(R, M) \cong M/[R, M]$.

Definition 3. Consider the cosimplicial k-module $[n] \mapsto \mathrm{Hom}_k(R^{\otimes n}, M)$ by

$$\delta_i(f)(r_0, r_1, \ldots, r_n) = \begin{cases} r_0f(r_1, \ldots, r_n) & \text{if } i = 0 \\ f(r_0, \ldots, r_{i-1}r_i, \ldots, r_n) & \text{if } 0 < i < n \\ f(r_0, \ldots, r_{n-1})r_n & \text{if } i = n \end{cases}$$

$$(\sigma_i f)(r_1, \ldots, r_{n-1}) = f(r_1, \ldots, r_{i-1}, 1, r_i, \ldots, r_{n-1}).$$

The *Hochschild cohomology* $\mathrm{HH}^*(R, M)$ of R with coefficients in M is defined to be the k-modules

$$\mathrm{HH}^n(R, M) = H_nC(\mathrm{Hom}_k(R^{\otimes n}, M)),$$

where $C(\mathrm{Hom}_k(R^{\otimes n}, M))$ is the associated chain complex with $\delta = \sum (-1)^i \delta_i$.

Proposition 4. $\mathrm{HH}^0(R, M) \cong \{m \in M : rm = mr \ \forall r \in R\}$.

Definition 5. The *enveloping algebra* of R is defined by $R^e = R \otimes R^{op}$.

Theorem 6. *$\mathrm{HH}(R, M) \cong \mathrm{Tor}_*^{R^e/k}(M, R)$, $\mathrm{HH}^*(R, M) \cong \mathrm{Ext}^*_{R^e/k}(R, M)$.*

Proposition 7. *If*

$$0 \to M_0 \to M_1 \to M_2 \to 0$$

is a k-split exact sequence of bimodules, then there is a long exact sequence

$$\cdots \xrightarrow{\partial} \mathrm{HH}_i(R, M_0) \to \mathrm{HH}_i(R, M_1) \to \mathrm{HH}_i(R, M_2) \xrightarrow{\partial} \mathrm{HH}_{i-1}(R, M_0) \to \cdots.$$

Remark 8. If R is flat as a k-module (this is always the case when k is a field), then $\mathrm{HH}(R, M) \cong \mathrm{Tor}_*^{R^e}(M, R)$ and $\mathrm{HH}^*(R, M) \cong \mathrm{Ext}^*_{R^e}(R, M)$.

Example 9. Let $T = T(V)$ be the tensor algebra of a k-module V, and let M be a $T-T$-bimodule. Then $\mathrm{HH}_i(T, M) = 0$ for $i \neq 0, 1$ and there is an exact sequence

$$0 \to \mathrm{HH}_1(T, M) \to M \otimes V \xrightarrow{b} M \to \mathrm{HH}_0(T, M) \to 0$$

where $b(m \otimes v) = mv - vm$.

When $M = T$, we have

$$\mathrm{HH}_0(T, T) = k \oplus \bigoplus_{i=1}^{\infty} (V^{\otimes i})_\sigma,\ \mathrm{HH}_1(T, T) = \bigoplus_{i=1}^{\infty} (V^{\otimes i})^\sigma,$$

where $\sigma(v_1 \otimes \cdots \otimes v_j) = v_j \otimes v_1 \otimes \cdots \otimes v_{j-1}$.

Example 10. If $R = k[x_1, \ldots, x_n]$, then

$$\mathrm{HH}_i(R, R) \cong \mathrm{HH}_i(R, R) \cong \wedge^i(R^n).$$

Example 11. If $R = k[x]/(x^{n+1})$, then $\mathrm{HH}_i(R, R)$ and $\mathrm{HH}^i(R, M)$ are 2-periodic for $i \geq 1$. In particular, when $\frac{1}{n+1} \in R$ we have $\mathrm{HH}_i(R, R) \cong \mathrm{HH}^i(R, R) \cong R/(x^n R) \cong k[x]/(x^n)$ for all $i \geq 1$.

Theorem 12 (Change of ground rings)**.** *Let $k \to \ell$ be a commutative ring homomorphism. Then*

$$\mathrm{HH}_*^k(R, M) \cong \mathrm{HH}_*^\ell(R \otimes_k \ell, M), \quad \mathrm{HH}_k^*(R, M) \cong \mathrm{HH}_\ell^*(R \otimes_k \ell, M).$$

Theorem 13 (Change of rings)**.**

(1) *(Product) If M' is another R'-R'-bimodule, then*

$$\mathrm{HH}_*(R \times R', M \times M') \cong \mathrm{HH}_*(R, M) \oplus \mathrm{HH}_*(R', M')$$

(2) *(Flat base change) If T is a ring and a flat R-module, then*

$$\mathrm{HH}_*(T, T \otimes_R M \otimes_R T) \cong T \otimes_R \mathrm{HH}_*(R, M)$$

(3) *If S is a central multiplicative set in R, then*

$$\mathrm{HH}_*(S^{-1}R, S^{-1}R) \cong \mathrm{HH}_*(R, S^{-1}R) \cong S^{-1}\mathrm{HH}(R, R)$$

Proposition 14. *Let M, N be left R-modules. Then $\mathrm{Hom}_k(M, N)$ becomes an R-R-bimodule and*

$$\mathrm{HH}^n(R, \mathrm{Hom}_k(M, N)) \cong \mathrm{Ext}^n_{R/k}(M, N).$$

Note that the kernel of the map $\delta : \mathrm{Hom}_k(R, M) \to \mathrm{Hom}_k(R \otimes R, M)$ is the set of all k-linear functions $f : R \to M$ satisfying the identity

$$f(r_0 r_1) = r_0 f(r_1) + f(r_0) r_1.$$

On the other hand, the image of $d : M \to \mathrm{Hom}_k(R, M)$ is the set of $f_m(r) = rm - mr$. We call them the principal derivations and write $\mathrm{PDer_k(R, M)}$.

Proposition 15. $\mathrm{HH}^1(R, M) = \mathrm{Der}_k(R, M)/\mathrm{PDer}_k(R, M)$.

Definition 16. Assume that R is commutative. the Kähler differentials of R over k is the R-module $\Omega_{R/k}$ defined by

$$\Omega_{R/k} = R\{dr : r \in R\}/I$$

where I is the submodule generated by $d\alpha = 0$ for all $\alpha \in k$, $d(r_0 + r_1) = d(r_0) + d(r_1)$ and $d(r_0 r_1) = r_0(dr_1) + (dr_0) r_1$.

Proposition 17. *Let R be a commutative k-algebra, and M a right R-module. Making M into an R-R bimodule by the rule $rm = mr$, we have natural isomorphisms $\mathrm{HH}_0(R,M) \cong M$ and $\mathrm{HH}_i(R,M) \cong M \otimes_R \Omega_{R/k}$. In particular,*

$$\mathrm{HH}_1(R,R) \cong \Omega_{R/k}.$$