LECTURE 4 - SIMPLICIAL HOMOTOPY

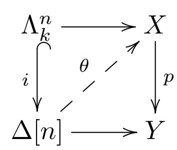
In this lecture, $\Delta[n] = \Delta[n]^s \in s\mathscr{S}et$ is the standard n-simplex in the category of simplicial sets.

Definition 1. The boundary of the $\Delta[n] \in s\mathscr{S}et$ is defined by

$$(\partial\Delta[n])_q = \begin{cases} \Delta[n]_q & q \leq n-1 \\ \text{iterated degeneracies of elements of the above} & q \geq n. \end{cases}$$

The k^{th} horn $\Lambda^n_k \subset \Delta[n]$ is the subcomplex of $\Delta[n]$ which is generated by all faces $d_j(\iota_n)$ except the k^{th} face $d_k(\iota_n)$. Here $\iota_n = id \in \Delta([n],[n])$ is the nondegenerate n-simplex of $\Delta[n]$.

Definition 2. A map $p : X \to Y$ of simplicial set is said to be a *(Kan) fibration* if for every commutative simplicial set homomorphisms



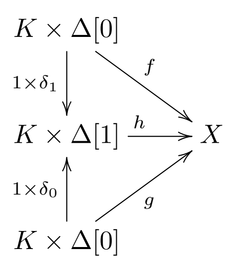
There is a map $\theta : \Delta[n] \to X$ making the diagram commute.

A *fibrant* simplicial set (or *Kan complex*) is a simplicial set Y such that the canonical map $Y \to *$ is a fibration.

Proposition 3. *The simplicial set Y is a Kan complex if and only if for each n-tuple of $(n-1)$-simplicies $(y_0, \dots, y_{k-1}, y_{k+1}, \dots, y)$ of Y such that $d_i y_j = d_{j-1} y_i$ if $i < j, i, j \neq k$, there is an n-simplex y such that $d_i y = y_i$ for $i \neq k$.*

Example 4. The singular complexes $S_\bullet(X)$ of a topological spaces X, the classifying space BG of groups G and the underlying simplicial sets of simplicial groups are all Kan complex.

Definition 5. Let $f, g : K \to X$ be simplicial maps. We say that there is a simplicial homotopy $f \xrightarrow{\simeq}$ from f to g if there is a commutative diagram



The map h is called a *homotopy.*

The homotopy relation may fail to be an equivalence relation in general.

Lemma 6. *If X is a Kan complex, then simplicial homotopy of vertices $x : \Delta[0] \to X$ of X is an equivalence relation.*

Theorem 7. *Suppose X is fibrant and $L \subset K$ is inclusion of simplicial sets. Then*

(1) *homotopy of maps $K \to X$ an equivalence relation, and*
(2) *homotopy of maps $K \to X$(rel L) is an equivalence relation.*

Definition 8. Let X be a fibrant simplicial set and let $v \in X_0$ be a vertex of X. Define $\pi_n(X, v) : n \geq 1$, to be the set of homotopy classes of maps $\alpha : \Delta[n] \to X$ (rel $\partial\Delta[n]$) for maps α which fit into diagrams

$$\begin{array}{ccc} \Delta[n] & \xrightarrow{\alpha} & X \\ \uparrow & & \uparrow{\scriptstyle v} \\ \partial\Delta[n] & \longrightarrow & \Delta[0] \end{array}$$

Suppose $\alpha, \beta : \Delta[n] \to X$ represent elements of $\pi_n(X, v)$. Then the simplices

$$\begin{cases} v_i = v, & 0 \leq i \leq n-2 \\ v_{n-1} = \alpha, & \\ v_{n+1} = \beta, & \end{cases}$$

satisfy $d_i v_j = d_{j-1} v_i$ if $i < j$ and $i, j \neq n$, since all faces of all simplices v_i map through the vertex v. Thus, the v_i determine a simplicial set map

$$(v_0, \ldots, v_{n-1}, , v_{n+1}) : \Lambda^{n+1}_n \to X,$$

and there is an extension

$$\begin{array}{ccc} \Lambda^{n+1}_n & \xrightarrow{(v_0,\ldots,v_{n-1},,v_{n+1})} & X \\ \downarrow & \nearrow{\scriptstyle \omega} & \\ \Delta[n+1] & & \end{array}$$

The assignment

$$([\alpha], [\beta]) \to [d_n \omega]$$

gives a well-defined pairing

$$m : \pi_n(X, v) \times \pi_n(X, v) \to \pi_n(X, v).$$

Theorem 9. *With these definitions, $\pi_n(X, v)$ is a group for $n \geq 1$ and abelian if $n \geq 2$.*

Definition 10. Let X, Y be simplicial sets. The *function complex* $\mathbf{Hom}(X, Y)$ is the simplicial set defined by

$$\mathbf{Hom}(X, Y)_n = \hom(X \times \Delta[n], Y).$$

Suppose that $p : X \to Y$ is a Kan fibration between Kan complexes X, Y, and F is the fiber over a vertex $* \in Y$ in the sense that the square

$$\begin{array}{ccc} F & \xrightarrow{i} & X \\ \downarrow & & \downarrow{\scriptstyle p} \\ \Delta[0] & \xrightarrow[*]{} & Y \end{array}$$

is a pullback. Suppose that v is a vertex of F and that $\alpha : \Delta[n] \to Y$ represents an element of $\pi_n(Y, *)$. Then in the diagram

$$\begin{array}{ccc} \Lambda^n_0 & \xrightarrow{(,v,\ldots,v)} & X \\ \big\downarrow & \overset{\theta}{\nearrow} & \big\downarrow p \\ \Delta[n] & \xrightarrow[\alpha]{} & Y \end{array}$$

The element $[d_0\theta] \in \pi_{n-1}(F, v)$ is independent of the choice of θ and representative of $[\alpha]$.

Definition 11. The resulting function

$$\partial : \pi_n(Y, *) \to \pi_{n-1}(F, v)$$

is called the *boundary map*.

Theorem 12. *The boundary map $\partial : \pi_n(Y, *) \to \pi_{n-1}(F, v)$ is a group homomorphism if $n > 1$, and the sequence*

$$\cdots \to \pi_n(F, v) \xrightarrow{i_*} \pi_n(X, v) \xrightarrow{p_*} \pi_n(Y, *) \xrightarrow{\partial} \pi_{n-1}(F, v) \to \cdots$$
$$\cdots \xrightarrow{p_*} \pi_1(Y, *) \xrightarrow{\partial} \pi_0(F) \xrightarrow{i_*} \pi_0(X) \xrightarrow{p_*} \pi_0(Y)$$

is exact in the sense the the kernel equals image everywhere.

Next we introduce some constructions in the category of simplicial sets.

Proposition 13. *There is an adjunction*

$$\hom(K, \mathbf{Hom}(X, Y)) \xrightarrow{\cong} \hom(X \times K, Y).$$

Definition 14. For a Kan complex X and a vertex $*$ of X, the *path space* PX is defined by the pullback diagram

$$\begin{array}{ccc} PX & \longrightarrow & \mathbf{Hom}(\Delta[1], X) \\ \big\downarrow i_X & & \big\downarrow (\delta_0)^* \\ \Delta[0] & \xrightarrow[*]{} & \mathbf{Hom}(\Delta[0], X) \cong X \end{array} .$$

Proposition 15. *The map $\pi : PX \to X$ given by the composite*

$$PX \xrightarrow{i_X} \mathbf{Hom}(\Delta[1], X) \xrightarrow{(\delta_1)^*} \mathbf{Hom}(\Delta[0], X) \cong X$$

is a fibration and PX is also fibrant.

$\pi_i(PX, v)$ *is trivial for $i \geq 0$ and all vertices v.*

Definition 16. The loop space ΩX is defined to be the fiber of $\pi : PX \to X$ over the base point $*$.

A simplex of ΩX can be identified with a simplicial map $f : \Delta[n] \times \Delta[1] \to X$ such that the restriction of f to $\Delta[n] \times \partial\Delta[1]$ maps into $*$.