# LECTURE 1 - COFIBER SEQUENCES

The goal of this lecture is to introduce the cofiber sequence of a pair and relate it to the long exact sequence of the cohomology groups.

Let $\mathscr{T}$ be the category of (compactly generated) pointed spaces.

**Definition 1.** For $X, Y \in \mathscr{T}$, the *smash product* $X \wedge Y$ is defined by
$$X \wedge Y = X \times Y / X \vee Y.$$

**Definition 2.** For $X \in \mathscr{T}$, we define the *cone* on $X$ to be $CX = X \wedge I$, and the *suspension* of $X$ to be $\Sigma X = X \wedge S^1$.

**Theorem 3.** *Let $F(X, Y)$ be the space of based maps from $X$ to $Y$. We have a natural homeomorphism of based spaces*
$$F(X \wedge Y, Z) \cong F(X, F(Y, Z))$$

**Example 4.** We define the *loop space* of $X$ to be $\Omega X = F(S^1, X)$. Then we have the adjunction
$$F(\Sigma X, Y) \cong F(X, \Omega Y).$$
Passing to $\pi_0$, this gives that
$$[\Sigma X, Y] = [X, \Omega Y].$$
If we let $X = S^n$, then this further gives that
$$\pi_{n+1} Y = \pi_n \Omega Y.$$

**Definition 5.** For an abelian group $A$ and an integer $n \geq 0$, we define the *Eilenberg-MacLane* space $K(A, n)$ to be a based CW complex such that
$$\pi_i(K(\pi, n)) \cong \begin{cases} A, & i = n, \\ 0, & i \neq n. \end{cases}$$
If $n = 0$ or 1, the group $A$ is allowed to be non-abelian.

**Example 6.** $K(\mathbb{Z}, 1) = S^1$. $K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$. $K(\mathbb{Z}/2, 1) = \mathbb{R}P^\infty$.

**Theorem 7.** *The Eilenberg-MacLane spaces are unique up to homotopy equivalences.*

**Corollary 8.** $K(A, n) \simeq \Omega K(A, n+1)$.

**Theorem 9.** $H^n(X, A) \cong [X, K(A, n)]$.

**Definition 10.** For a based map $f : X \to Y$, define the *homotopy cofiber* $Cf$ to be
$$Cf = Y \cup_f CX = Mf / j(X)$$
where $j : X \to Mf$ sends $x \to (x, 1)$.

The inclusion $i : Y \to Cf$ is a cofibration since it is the pushout of $f$ and the cofibration $X \to CX$ that sends $x$ to $(x, 0)$. Let
$$\pi : Cf \to Cf/Y \cong \Sigma X$$
be the quotient map.

**Definition 11.** The sequence

$$X \xrightarrow{f} Y \xrightarrow{i} Cf \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma i} \Sigma Cf \xrightarrow{-\Sigma \pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \Sigma^2 Y \to \cdots$$

is called the *cofiber sequence generated by the map* $f$; here

$$(-\Sigma f)(x \wedge t) = f(x) \wedge (1-t).$$

**Theorem 12.** *For any based space* $Z$*, the induced sequence*

$$\cdots \to [\Sigma^2 X, Z] \to [\Sigma Cf, Z] \to [\Sigma Y, Z] \to [\Sigma X, Z] \to [Cf, Z] \to [Y, Z] \to [X, Z]$$

*is an exact sequence of pointed sets, or of groups to the left of* $[\Sigma X, Z]$*, or of Abelian groups to the left of* $[\Sigma^2 X, Z]$*.*

**Reading Homework 13.** Read Chapter 8 Section 4 of [1] for the proof of the theorem.

**Corollary 14.** *In the theorem, let* $Z = K(A, n)$*. then we have*

$$\cdots \to H^{n-1}(Cf) \to H^{n-1}(Y) \to H^{n-1}(X) \to H^n(Cf) \to H^n(Y) \to H^n(X).$$

*We can actually extend to the right by choosing bigger* $n$ *(why?).*

## References

[1] J. P. May. *A concise course in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1999.